Quantitative Finance

ISSN: 1469-7688 (Print) 1469-7696 (Online) Journal homepage: http://www.tandfonline.com/loi/rquf20

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To cite this article: Mustafa Ç. Pinar (2013) Buyer's quantile hedge portfolios in discrete-time trading, Quantitative Finance, 13:5, 729-738, DOI: 10.1080/14697688.2010.538075

To link to this article: http://dx.doi.org/10.1080/14697688.2010.538075

Published online: 22 Mar 2011.

Quantitative Finance, 2013
Vol. 13, No. 5, 729–738, http://dx.doi.org/10.1080/14697688.2010.538075

Buyer's quantile hedge portfolios in discrete-time trading

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(*Received 21 April 2009; in final form 29 October 2010*)

The problem of quantile hedging for American claims is studied from the perspective of the buyer of a contingent claim by minimizing the 'expected failure ratio'. After a general study of the problem in infinite-state spaces, we pass to finite dimensions and examine the properties of the resulting finite-dimensional optimization problems. In finite-state probability spaces we obtain a bilinear programming formulation that admits an exact linearization using binary exercise variables. Numerical results with S&P 500 index options demonstrate the computational viability of the formulations.

Keywords: Asset pricing; Optimization; Risk management; American style derivative securities

1. Introduction

A fundamental problem of financial economics is the pricing of financial instruments called 'contingent claims'. When an arbitrage-free financial market is not complete, it is well-known that there exists a set of 'risk-neutral' probability measures that make the (discounted) prices of traded instruments martingales. An important feature of the set of risk-neutral measures is that the value of the cheapest portfolio to dominate the pay-off at maturity of a contingent claim coincides with the maximum expected value of the (discounted) pay-off of the claim with respect to this set. This value, called the 'super-hedging price', allows the seller to assemble a portfolio that achieves a value at least as large as the pay-off to the claim holder at the maturity date of the claim in all non-negligible events. By a similar reasoning, the largest price that a potential buyer is willing to disburse to acquire a contingent claim is called a 'sub-hedging price' (or lower hedging price), which is equal to the value of the most precious portfolio that is dominated by the contingent claim pay-off at maturity. (If the claim is attainable, then the smallest price to super-hedge and the largest price to sub-hedge are equal to the hedging price, and the expected value does not depend on the chosen risk-neutral measure, so the previous statement still applies.) The super-hedging price is the natural price to be asked by the writer of a contingent claim and, together with the bid price obtained by considering the analogous problem from the point of view of the buyer, it constitutes an interval that is sometimes called the 'no-arbitrage price interval' for the claim in question.

A writer may not always ask for the whole super-hedging price to 'sell' a claim with pay-off F_T (see, e.g., Föllmer and Schied 2004, chapters 7 and 8 for a discussion and examples showing that the super-hedging price may be too high). On the other hand, some economic considerations such as pre-existing endowments or liabilities may induce a buyer to pay a larger price than the sub-hedging price to acquire the claim. It may also be the case that the no-arbitrage buyer price, i.e. the sub-hedging price, may be too low to be interesting for any potential seller. In such a case, neither buyer nor seller will be able to set up sub-hedging or super-hedging portfolios, which implies that they will face a positive probability of 'falling short', i.e. for the writer his/her portfolio will take values V_T smaller than those of the claim on a non-negligible event, and for the buyer his/her portfolio will take values V_T larger than those of the claim on a non-negligible event. Thus, the writer and the buyer will need to choose hedging strategies according to some optimality criterion to be decided. One such criterion that has been widely studied comes from the idea of quantile hedging, which consists of choosing a hedge portfolio that minimizes the probability of a shortfall in the case of a writer. The problem has been studied in several papers (Spivak and Cvitanić 1999, Föllmer and Leukert 2000, Nakano 2003, 2004, Rudloff 2007, 2009) in discrete time

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and infinite state spaces or in continuous time, almost exclusively—to the best of this author's knowledge—from the viewpoint of a writer of a European claim. Pérez-Hernández (2007) studied the problem of quantile hedging for American contingent claims in an infinite-state space setting from the perspective of the writer of the claim.

The purpose of this paper is to study quantile hedging portfolio strategies from a buyer's perspective under discrete-time trading in incomplete markets for American claims. We analyse the problem both in infinite- and finite-state spaces. In the latter case, we formulate the associated optimization problems as problems that can be treated numerically by available software, and investigate their properties. The resulting bilinear programming formulations for lower quantile hedging of American contingent claims are new, and appear for the first time in the present paper, to the best of the author's knowledge.

The rest of the paper is organized as follows. In section 2 we recall briefly the quantile hedge problem from a writer's point of view. Section 3 is devoted to the quantile hedge problems of American contingent claims in general, and section 4 to the same problems in finite-dimensional spaces, from a buyer's perspective. An interesting auxiliary—but important in its own right—result allows us to obtain a compact, bilinear, continuous optimization formulation. Using the binary nature of variables that represent exercise strategies for the American claim, we obtain a linear mixed-integer programming formulation that is equivalent to the bilinear continuous formulation. Finally, section 5 is devoted to the numerical testing of the formulations of the paper using data from S&P 500 options. It appears that the linearized version of the bilinear formulation, while giving rise to large optimization problems, may be processed numerically by available software.

A common criticism leveled against the criterion of quantile hedging is that it ignores the magnitude of the losses (Föllmer and Schied 2004). In response to this criticism, several authors have studied the criterion of expected shortfall minimization (see, e.g., Nakano (2003), Föllmer and Schied (2004) and Rudloff (2007, 2009)). We will investigate this criterion in the spirit of the present paper, that is, from a buyer's perspective, in a subsequent work.

2. Background on writer's quantile hedge for European claims

In the present paper, we work in a financial market $\mathcal{M}=(\Omega,\mathcal{F},\mathbb{P},\mathbb{T},S,\{\mathcal{F}_t\}_{t\in\mathbb{T}})$ with discrete-time trading over the time set $\mathbb{T}=\{0,1,\ldots,T\}$ and where $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}\}_{t\in\mathbb{T}})$ is a complete filtered probability space, and $S=\{S_t\}_{t\in\mathbb{T}}$ is an $\mathbb{R}^2_+$ asset price process over the time set $\mathbb{T}$ adapted to the filtration $\{\mathcal{F}\}_{t\in\mathbb{T}}$. We assume without loss of generality that the first component of S is the numéraire security, i.e. $S^0_t=1$ for all $t\in\mathbb{T}$. Let $\mathcal{Q}$ be the set of equivalent martingale measures in the arbitrage-free (not necessarily complete) market $\mathcal{M}$. For the rest of the paper we make the following blanket assumption.

Assumption 2.1: *The market is arbitrage free, i.e. the set $\mathcal{Q}$ is non-empty.*

Let us recall the problem of quantile hedging from the point of view of the writer of a contingent claim. The problem in the form that we will address was studied by Föllmer and Leukert (2000). The idea is the following. Fix a capital v smaller than the no-arbitrage price for the contingent claim H with (discounted with respect to the numéraire) pay-offs $\{H_t\}_{t\in\mathbb{T}}$, i.e. $v<\Pi^{\uparrow}(H)\equiv\sup_{\mathbb{Q}\in\mathcal{Q}}\mathbb{E}^{\mathbb{Q}}[H_T]$, and find a hedge policy that maximizes the probability of success, where the probability of success is the probability of the event that the value of an admissible portfolio strategy at the time of expiration of the claim is at least as large as the value of the claim. A self-financing trading strategy is called admissible if its value process is non-negative at the time of expiration of the contingent claim (Föllmer and Schied 2004, definition 8.1). Hence, the problem is to construct an admissible strategy $\bar{\xi}^*$ such that its value process V^* satisfies

$$\mathbb{P}[V^*_T\geq H]=\max\mathbb{P}[V_T\geq H], \tag{1}$$

where the maximum is searched for in the set of all admissible portfolio strategies satisfying

$$V_0\leq v. \tag{2}$$

The set $\{V_T\geq H\}$ is termed the *success set*. Föllmer and Schied (2004) show that the problem is guaranteed to admit a closed-form solution for complete markets, under a technical condition that may not always be verified (Föllmer and Leukert 2000). More precisely, let $\mathcal{Q}$ be a singleton, i.e. $\mathcal{Q}=\{\mathbb{Q}^*\}$. Assuming that A^* maximizes the probability $\mathbb{P}[A]$ among all sets $A\in\mathcal{F}_T$ satisfying the constraint

$$\mathbb{E}^{\mathbb{Q}^*}[H\mathbb{1}_A]\leq v,$$

the replicating strategy $\bar{\xi}^*$ of the knock-out option $H^*=H\mathbb{1}_{A^*}$ solves the above optimization problem. To construct the optimal set A^* using the Neyman–Pearson theory, an auxiliary measure Q^* is introduced, given by

$$\frac{\mathrm{d}Q^*}{\mathrm{d}\mathbb{Q}^*}=\frac{H}{\mathbb{E}^{\mathbb{Q}^*}[H]}.$$

The set A^* was explicitly pointed out by Föllmer and Schied (2004, chapter 8). If

$$Q^*[A^*]=\frac{v}{\mathbb{E}^{\mathbb{Q}^*}[H]},$$

then A^* maximizes $\mathbb{P}[A]$ among all sets $A\in\mathcal{F}_T$ satisfying the constraint

$$\mathbb{E}^{\mathbb{Q}^*}[H\mathbb{1}_A]\leq v,$$

and the replicating strategy of $H\mathbb{1}_{A^*}$ solves the problem (1)–(2). However, in general it may be impossible to find such a set A^* with probability under Q^* exactly equal to $v/\mathbb{E}^{\mathbb{Q}^*}[H]$. Hence, Föllmer and Leukert introduced an extended version of the problem where the so-called 'expected success ratio' is minimized.

For an admissible strategy ξ and its value process V we define the success ratio ψ_V of ξ by

$$\psi_V := \mathbb{1}_{\{V_T \geq H\}} + \frac{V_T}{H}\mathbb{1}_{\{V_T < H\}} = \frac{V_T}{H} \wedge 1.$$

For an amount $v < \Pi^{\uparrow}(H)$, the writer's problem consists of searching among all admissible strategies, with initial endowment equal to at most v, for one that maximizes the expected success ratio. In other words, the problem of interest to the writer is problem [WECP] over variables ξ that are admissible adapted portfolio strategies ξ (and their value processes V)

$$\begin{aligned} \sup \quad & \mathbb{E}^{\mathbb{P}}[\psi_V], \\ \text{s.t.} \quad & V_0 \leq v. \end{aligned}$$

The set $\{\psi_V = 1\}$ coincides with the success set $\{V_T \geq H\}$ of V. Instead of solving the above problem, Föllmer and Leukert (2000) and Föllmer and Schied (2004) advocate solving the auxiliary problem

$$\begin{aligned} \sup \quad & \mathbb{E}^{\mathbb{P}}[\psi], \\ \text{s.t.} \quad & \sup_{\mathbb{Q}\in\mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[H\psi] \leq v, \\ & \psi \in [0,1] \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

They show that a solution ψ^* exists and the value process of a super-hedging strategy for the modified claim $H\psi^*$ leads to the optimal success set corresponding to initial capital v. The details can be found in Föllmer and Schied (2004, chapter 8). The above ideas were extended to the case of American contingent claims from the viewpoint of the writer by Pérez-Hernández (2007).

3. Quantile hedging: Buyer's view for American claims

The study of American contingent claims poses an additional challenge due to the presence of the early exercise possibility. For an in-depth treatment of American contingent claims in discrete time, our desktop reference is Föllmer and Schied (2004). One common way to describe exercise strategies of American claims is by stopping times. These are functions $\tau : \Omega \to \{0, \ldots, T\} \cup \{+\infty\}$ such that $\{\omega \in \Omega \mid \tau(\omega) = t\} \in \mathcal{F}_t$, for each $t = 0, \ldots, T$. Let $\mathcal{T}$ denote the set of stopping times. It is well-known that the buyer lower hedging price of an American claim with pay-off process $C = \{C_t\}_{t\in\mathbb{T}}$ is given by

$$\Pi^{\downarrow}(C) = \sup_{\tau\in\mathcal{T}} \inf_{Q\in\mathcal{Q}} \mathbb{E}^{Q}[C_\tau].$$

We shall refer to $\Pi^{\downarrow}(C)$ as the sub-hedging price for American claim C. Chalasani and Jha (2001) show that $\Pi^{\downarrow}(C)$ can be obtained as the optimal value of the following optimization problem:

$$\max_{\xi\in\Xi}\{-V_0(\xi) \mid \exists\tau \in \mathcal{T} \text{ s.t. } V_\tau(\xi) + C_\tau \geq 0\},$$

where Ξ represents the set of self-financing portfolio strategies ξ. Assuming ξ^* is an optimal portfolio strategy and τ^* an optimal exercise rule, the buyer borrows the amount $V_0(\xi^*)$ at time 0 to pay the seller for the contingent claim, and acquires the claim. At the time τ^* of exercise of the claim, the buyer repays his/her debt incurred at time 0. We refer to the optimal portfolio strategy of the buyer as a 'sub-hedging strategy'. A sub-hedging strategy ξ^* has the property that $V_t(\xi^*) \leq C_t$ on $\{C_t > 0\}$ for all $t \in \mathbb{T}$, and $V_T \geq 0$ (Chalasani and Jha 2001, Föllmer and Schied 2004, Pennanen and King 2006).

Following Pérez-Hernández (2007), let the American contingent claim $C = \{C_t\}_{t\in\mathbb{T}}$ be a given non-negative adapted and $\mathbb{P}$-integrable process. For a portfolio strategy ξ we define the 'failure ratio' process ψ^ξ of ξ by

$$\psi_t^\xi := \mathbb{1}_{\{V_t(\xi)\leq C_t\}} + \frac{V_t(\xi)}{C_t}\mathbb{1}_{\{V_t(\xi) > C_t\}} = \frac{V_t(\xi)}{C_t} \vee 1.$$

For an amount $v > \Pi^{\downarrow}(C)$, the buyer's problem consists of searching among all self-financing portfolio strategies, with initial endowment equal to at least v, for one that minimizes the maximum of the *expected failure ratio* over all stopping times. In other words, the problem of interest to the buyer is problem ACP (American Claim Problem)

$$\begin{aligned} \inf \quad & \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{P}}[\psi_\tau^\xi], \\ \text{s.t.} \quad & V_0(\xi) \geq v. \end{aligned}$$

We define by R the set of $[1, \infty]$-valued adapted processes, i.e.

$$\mathsf{R} = \{\psi = \{\psi_t\}_{t\in\mathbb{T}} : \psi_t \in [1,\infty], \text{ and } \mathcal{F}_t\text{-measurable}, \forall t \in \mathbb{T}\}.$$

For the American claim C we define the subset R_0 of R

$$\mathsf{R}_0 = \left\{\psi \in \mathsf{R} : \inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau \psi_\tau] \geq v\right\}.$$

We are interested in solving the problem [PACP] as a proxy to [ACP]

$$\begin{aligned} \inf \quad & \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{P}}[\psi_\tau], \\ \text{s.t.} \quad & \inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau \psi_\tau] \geq v, \\ & \psi \in [1,\infty] \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

As an elementary observation note that, if $\hat{\psi}_t$ solves [PACP], every process $\bar{\psi} \in \mathsf{R}_0$ with $\bar{\psi}_t \leq \hat{\psi}_t$, $\forall t \in \mathbb{T}$, also solves [PACP].

Theorem 3.1: *Let the infimum in PACP be finite. Then we have the following.*

(1) *The problem [PACP] has an optimal solution, i.e. there exists a $\hat{\psi} \in \mathsf{R}_0$ with*

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{P}}[\hat{\psi}_\tau] = \min_{\psi\in\mathsf{R}_0}\left(\sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{P}}[\psi_\tau]\right).$$

(2) *For an optimal solution $\hat{\psi}$ of [PACP], a sub-hedging strategy $\hat{\xi}$ for the adjusted American claim $\hat{C} = C \cdot \hat{\psi}$ is a solution of problem ACP.*

Proof: For part 1, observe that in [PACP] the functional $f(\psi) \equiv \mathbb{E}^{\mathbb{P}}[\hat{\psi}_\tau]$ is a linear functional in ψ_τ for fixed τ. Since the supremum of an arbitrary collection of linear functionals is convex and piecewise linear (Van Tiel 1984), the objective function in [PACP] is a convex functional in ψ. Now fix $\lambda > \inf_{\psi \in \mathsf{R}_0} f$. Since the level set $\mathrm{lev}(f, \lambda) = \{\psi \mid f(\psi) \leq \lambda\}$ is convex, and closed (Van Tiel 1984), the set $\mathrm{lev}(f, \lambda) \cap \{\psi \mid \psi \in [1, \infty] \ \mathbb{P}\text{-a.s.}\}$ is closed and bounded, and hence compact using the assumption that $\inf_{\psi \in \mathsf{R}_0} f > -\infty$. On the other hand, the set $\{\psi \mid \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau \psi_\tau] \geq v\}$ is closed, hence its intersection with $\mathrm{lev}(f, \lambda) \cap \{\psi \mid \psi \in [1, \infty] \ \mathbb{P}\text{-a.s.}\}$ is a compact set. Since PACP is equivalent to minimizing f over $\mathrm{lev}(f, \lambda) \cap \{\psi \mid \psi \in [1, \infty] \ \mathbb{P}\text{-a.s.}\} \cap \{\psi \mid \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau \psi_\tau] \geq v\}$, a compact set by the previous assertion, f attains its infimum over this set by the Weierstrass' theorem. This proves part 1.

For part 2, let ξ be a self-financing portfolio strategy with $V_0(\xi) \geq v$ and $V_T \geq 0$, and $\tau \in \mathcal{T}$ be a stopping time. Using Doob's Stopping Theorem (Föllmer and Schied 2004, theorem 6.17) on the $\mathbb{Q}$-martingale $V(\xi)$ we have that

$$\mathbb{E}^{\mathbb{Q}}[C_\tau \psi_\tau^\xi] = \mathbb{E}^{\mathbb{Q}}[V_\tau(\xi) \vee C_\tau] \geq \mathbb{E}^{\mathbb{Q}}[V_0(\xi)] \geq v,$$

hence $\psi^\xi \in \mathsf{R}_0$, and as a feasible point of [ACFP] we obtain

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\psi_\tau^\xi] \geq \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\hat{\psi}_\tau]. \tag{3}$$

Now let us consider a sub-hedging strategy $\hat{\xi}$ for the adjusted American claim $\hat{C} \equiv C\hat{\psi}$. Using the corresponding failure ratio $\psi^{\hat{\xi}}$ we have

$$C_t \cdot \psi_t^{\hat{\xi}} = C_t \vee V_t(\hat{\xi}) \leq C_t \vee (C_t \hat{\psi}) = C_t \hat{\psi}.$$

Therefore, $\psi_t^{\hat{\xi}}$ is dominated by $\hat{\psi}_t$ on the set $\{C_t > 0\}$. Moreover, any failure ratio is equal to 1 on $\{C_t = 0\}$ following a reasoning similar to Pérez-Hernández (2007). Hence, we obtain

$$\psi_t^{\hat{\xi}} \leq \hat{\psi}_t, \quad \mathbb{P}\text{-a.s.}$$

Now, since every process $\bar{\psi}$ such that $\bar{\psi}_t \leq \hat{\psi}_t$ for all $t \in \mathbb{T}$ is also a solution to PACP we obtain that $\hat{\xi}$ solves [PACP]. Combined with (3), we have that $\sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\psi_\tau^{\hat{\xi}}] = \sup_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{P}}[\hat{\psi}_\tau]$ and $\hat{\xi}$ solves [ACP]. □

For an optimal ψ^*, the sub-hedging portfolio strategy for the scaled-up claim $C\psi^*$ is the optimal quantile hedge strategy corresponding to initial capital v. We next investigate the ramifications of this result in finite-state markets.

4. Buyer's problem for American claims in finite-state markets

In finite-state markets, we can transform PACP into a finite-dimensional optimization problem that can be processed numerically by existing optimization algorithms and software.

4.1. *The finite-state market*

Now we assume Ω has a finite number of atoms, i.e. $\Omega = \{\omega_1, \ldots, \omega_K\}$. More precisely, we assume as in King (2002) that security prices and other payments are discrete random variables supported on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose atoms are sequences of real-valued vectors (asset values) over discrete time periods $t \in \mathbb{T} = \{0, 1, \ldots, T\}$. The market evolves as a discrete, non-recombinant scenario tree in which the partition of probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one-to-one with nodes $n \in \mathcal{N}_t$ at level t in the tree. The set $\mathcal{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. While not needed in the finite probability setting, the σ-algebras $\mathcal{F}_t$ generated by the partitions $\mathcal{N}_t$ are such that $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_t \subset \mathcal{F}_{t+1}$ for all $0 \leq t \leq T-1$ and $\mathcal{F}_T = \mathcal{F}$. A stochastic process is said to be $(\mathcal{F}_t)_{t=0}^T$-adapted if, for each $t = 0, \ldots, T$, the outcome of the process only depends on the element of $\mathcal{F}_t$ that has been realized at stage t. Similarly, a decision process is said to be $(\mathcal{F}_t)_{t=0}^T$-adapted if, for each $t \in \mathbb{T}$, the decision depends on the element of $\mathcal{F}_t$ that has been realized at stage t. In the scenario tree, every node $n \in \mathcal{N}_t$ for $t = 1, \ldots, T$ has a unique parent denoted $\pi(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, 1, \ldots, T-1$, has a non-empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$. We denote the set of all nodes in the tree by $\mathcal{N}$. The set $\mathcal{A}(n)$ denotes the collection of ascendant nodes or the unique path leading to node n (including itself) from node 0 and the set $\mathcal{D}(n)$ denotes the collection of descendant nodes of node n including itself. The probability distribution $\mathbb{P}$ is obtained by attaching positive weights p_n to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. For each non-leaf (intermediate level) node in the tree, i.e. for all $\ell \in \mathcal{N} \backslash \mathcal{N}_T$, the probability is determined recursively by

$$p_\ell = \sum_{m \in \mathcal{C}(n)} p_m, \quad \forall n \in \mathcal{N}_t, \ t = T-1, \ldots, 0.$$

Hence, each non-leaf node has a probability mass equal to the combined mass of its child nodes.

A stochastic process $\{X_t\}$ is a time-indexed collection of random variables such that each X_t is a random variable with realizations X_n for all $n \in \mathcal{N}_t$. The expected value of X_t is uniquely defined by the sum

$$\mathbb{E}^{\mathbb{P}}[X_t] := \sum_{n \in \mathcal{N}_t} p_n X_n.$$

The conditional expectation of X_{t+1} on $\mathcal{N}_t$ is given by the expression

$$\mathbb{E}^{\mathbb{P}}[X_{t+1} \mid \mathcal{N}_t] := \sum_{m \in \mathcal{C}(n)} \frac{p_m}{p_n} X_m.$$

The market consists, as in the previous section, of a bond and a single risky security with prices at node n given by the two-dimensional vector $S_n = (S_n^0, S_n^1)^T$. The number of shares of securities held by the investor in state (node) $n \in \mathcal{N}_t$ is denoted $\theta_n \in \mathbb{R}^2$. Therefore, to each state $n \in \mathcal{N}_t$ is associated the two-dimensional real vector θ_n. The value of the portfolio at state n is $S_n \cdot \theta_n$. We assume without loss of generality that prices at all nodes have been scaled so that $S_n^0 = 1$ for all $n \in \mathcal{N}$. The assumption of a single risky security can easily be relaxed, and the development of the paper can be repeated for multiple securities, *mutatis mutandis*.

Definition 4.1: If there exists a probability measure $\mathbb{Q} = \{q_n\}_{n \in \mathcal{N}_T}$ such that

$$S_t = \mathbb{E}^{\mathbb{Q}}[S_{t+1} \mid \mathcal{N}_t] \quad (t \leq T-1),$$

then the process $\{S_t\}$ is called a martingale under $\mathbb{Q}$, and $\mathbb{Q}$ is called a martingale probability measure for the process $\{S_t\}$.

We denote (as usual) by $\mathcal{Q}$ the set of all equivalent probability measures that make S a martingale over $[0, 1, \ldots, T]$.

4.2. *Lower hedging price for the buyer of an American claim*

Before embarking on this transformation we need an auxiliary result that is important in its own right and will let us formulate the pricing problem of the buyer as a linear program with all the nice duality theory attached to it.

We define the sets of exercise strategies

$$E = \left\{ e \mid e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted}, \ \sum_{t=0}^T e_t \leq 1 \text{ and } e_t \in \{0,1\} \ P\text{-a.s.} \right\},$$

$$\tilde{E} = \left\{ e \mid e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted}, \ \sum_{t=0}^T e_t \leq 1 \text{ and } e_t \geq 0 \ P\text{-a.s.} \right\}.$$

The relation $e_t = 1 \Leftrightarrow \tau = t$ defines a one-to-one correspondence between stopping times and decision processes $e \in E$. Now, let us recall that an arbitrage seeking buyer's problem for an American contingent claim C with payoffs C_n for all $n \in \mathcal{N}$, i.e. the computation of the sub-hedging price $\Pi^{\downarrow}(C)$ and the sub-hedging strategy, can be formulated as the following problem that we will refer as AP1 (Pennanen and King 2006, Camcı and Pınar 2009):

$$\begin{aligned}
&\max V, \\
&\text{s.t. } S_0 \cdot \theta_0 = C_0 e_0 - V, \\
&S_n \cdot (\theta_n - \theta_{\pi(n)}) = C_n e_n, \quad \forall n \in \mathcal{N}_t, 1 \leq t \leq T, \\
&S_n \cdot \theta_n \geq 0, \quad \forall n \in \mathcal{N}_T, \\
&\sum_{m \in \mathcal{A}(n)} e_m \leq 1, \quad \forall n \in \mathcal{N}_T, \\
&e_n \in \{0,1\}, \quad \forall n \in \mathcal{N}.
\end{aligned}$$

The optimal value of the variable V is the largest amount that a potential buyer is willing to disburse for acquiring a given American contingent claim C, i.e. it gives the sub-hedging price. The optimal values of the variables θ_n give the sub-hedging strategy for the American contingent claim C. The computation of the sub-hedging price via the above integer programming problem is carried out by the construction of an optimal (adapted) portfolio process that satisfies the requirements described in the previous section. Such a portfolio process replicates the proceeds from the contingent claim (if exercised) by self-financing transactions using the market-traded securities so as to avoid any terminal losses. The first constraint expresses the value of the portfolio process at time 0. The second set of constraints are the self-financing portfolio rebalancing constraints at each subsequent trading date, and each node of the scenario tree at that trading date. The third set of constraints ensure that the value of the portfolio process at time T (end of the horizon) is non-negative. The integer-valued variables e_n and related constraints represent the one-time exercise of the American contingent claim (see Pennanen and King 2006 for further details). Alternatively, we can view the buyer implementing a sub-hedging strategy as follows. He/she borrows the amount $\Pi^{\downarrow}(C)$ to acquire the claim, and tries to close the debt positions with self-financing transactions and proceeds from the exercise of the claim in subsequent time periods in all states of the market.

A linear programming relaxation of AP1 is the following problem AP2:

$$\begin{aligned}
&\max V, \\
&\text{s.t. } S_0 \cdot \theta_0 = C_0 e_0 - V, \\
&S_n \cdot (\theta_n - \theta_{\pi(n)}) = C_n e_n, \quad \forall n \in \mathcal{N}_t, 1 \leq t \leq T, \\
&S_n \cdot \theta_n \geq 0, \quad \forall n \in \mathcal{N}_T, \\
&\sum_{m \in \mathcal{A}(n)} e_m \leq 1, \quad \forall n \in \mathcal{N}_T, \\
&e_n \geq 0, \quad \forall n \in \mathcal{N}.
\end{aligned}$$

Theorem 4.2: *There exists an optimal solution to AP2 with $e_n \in \{0,1\}$, $\forall n \in \mathcal{N}$.*

The proof of this theorem is somewhat involved and published elsewhere (Camcı and Pınar 2009).

A direct consequence of the above result is the following. We denote by $\tilde{\mathcal{Q}}$ the set of all martingale measures (not necessarily equivalent to $\mathbb{P}$) that is the closure of $\mathcal{Q}$ in discrete-time finite-state markets (King 2002).

Assuming no arbitrage in the financial market, the buyer's price for American contingent claim F can be expressed as in the following theorem (see Pennanen and King 2006 and Camcı and Pınar 2009 for a proof).

Theorem 4.3:

$$\max_{\tau \in \mathcal{T}} \min_{\mathbb{Q} \in \tilde{\mathcal{Q}}} \mathbb{E}^{\mathbb{Q}}[C_\tau] = \min_{\mathbb{Q} \in \tilde{\mathcal{Q}}} \max_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau]. \tag{4}$$

Corollary 4.4: $\inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau] = OPT(AP1) = OPT(AP2)$.

4.3. *Formulation of the buyer quantile hedge for American claims*

We now return to the problem [PACP]

$$\begin{aligned} \inf \quad & \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{P}}[\psi_\tau], \\ \text{s.t.} \quad & \inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau\psi_\tau] \geq v, \\ & \psi \in [1,\infty] \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

In finite-state markets we are facing the following problem:

$$\begin{aligned} \inf \quad & \sup_{e\in E} \mathbb{E}^{\mathbb{P}}[\psi\cdot e], \\ \text{s.t.} \quad & \inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{e\in E} \mathbb{E}^{\mathbb{Q}}[C\cdot\psi\cdot e] \geq v, \\ & \psi \in [1,\infty] \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

We shall now transform this problem step by step into an optimization problem that can be processed numerically by off-the-shelf optimization software.

Let us first treat the objective function

$$\inf_\psi \sup_{e\in E} \mathbb{E}^P[\psi\cdot e].$$

We first transform the inner maximization problem $\sup_{e\in E} \mathbb{E}^P[\psi\cdot e]$. Note that E is a discrete set. It is a set of binary vectors satisfying the one-time exercise inequalities $\sum_{m\in\mathcal{A}(n)} e_m \leq 1$, $\forall n\in\mathcal{N}_T$. On the other hand, the set $\tilde{E}$, which is the continuous relaxation of E, is the convex hull of the e's (Chalasani and Jha 2001, Pennanen and King 2006). This result follows from the fact that the matrix obtained from the one-time exercise inequalities above is an interval matrix (Schrijver 1986). Hence, it is a totally unimodular matrix, which implies that the set $\tilde{E}$ has binary extreme points coinciding with the binary vectors that constitute the set E. As a consequence of this property, optimizing a linear function over E and $\tilde{E}$ yields the same result, and since $\tilde{E}$ is compact the sup is attained. Therefore, we have

$$\max_{e\in E} \mathbb{E}^P[\psi\cdot e] = \max_{e\in\tilde{E}} \mathbb{E}^P[\psi\cdot e].$$

In other words, for fixed ψ we face the linear programming problem

$$\max_{e\in\tilde{E}} \left\{ \sum_{n\in\mathcal{N}} p_n\psi_n e_n \,\middle|\, \sum_{m\in\mathcal{A}(n)} e_m \leq 1 \ \forall n\in\mathcal{N}_T, e_n \geq 0 \ \forall n\in\mathcal{N} \right\}.$$

Now, attaching the non-negative Lagrange multipliers ζ_n to the constraints $\sum_{m\in\mathcal{A}(n)} e_m \leq 1$ $\forall n\in\mathcal{N}_T$ we obtain the Lagrange function

$$L(e_n,\zeta_n) = \sum_{n\in\mathcal{N}} p_n\psi_n e_n + \sum_{n\in\mathcal{N}_T} \zeta_n \left(\sum_{m\in\mathcal{A}(n)} e_m - 1 \right).$$

Maximizing the function L over non-negative e_n for $n\in\mathcal{N}$ we obtain the dual problem

$$\min_{\zeta_n\geq 0, n\in\mathcal{N}_T} \left\{ \sum_{n\in\mathcal{N}_T} \zeta_n \,\middle|\, \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} \zeta_m \geq p_n\psi_n \ \forall n\in\mathcal{N}_T \right\}.$$

Since the set $\tilde{E}$ is non-empty and compact, by the duality theorem of linear programming (Ben-Tal and Nemirovski 2001, theorem 1.3.2) we have

$$\begin{aligned} & \max_{e\in\tilde{E}} \mathbb{E}^P[\psi\cdot e] \\ & \quad = \min_{\zeta_n\geq 0, n\in\mathcal{N}_T} \left\{ \sum_{n\in\mathcal{N}_T} \zeta_n \,\middle|\, \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} \zeta_m \geq p_n\psi_n \ \forall n\in\mathcal{N}_T \right\}. \end{aligned}$$

This completes the first step of our transformation of the problem. Our problem now consists of minimizing with respect to variables $\psi_n \geq 1$ and $\zeta_n \geq 0$ the function

$$\sum_{n\in\mathcal{N}_T} \zeta_n$$

over the constraints

$$\sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} \zeta_m \geq p_n\psi_n, \quad \forall n\in\mathcal{N}_T,$$

and

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{e\in E} \mathbb{E}^{\mathbb{Q}}[C\cdot\psi\cdot e] \geq v. \tag{5}$$

We now transform the constraint (5) above. By theorem 4.2 the left-hand side of the inequality is equal to the optimal value of problem AP2 for the adjusted claim $C\psi$! That is, we have, for fixed ψ,

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{e\in E} \mathbb{E}^{\mathbb{Q}}[C\cdot\psi\cdot e] = \sup -S_0\cdot\theta_0 + C_0e_0\psi_0,$$

where the sup on the right-hand side is computed over the constraints

$$S_n\cdot(\theta_n - \theta_{\pi(n)}) = C_ne_n\psi_n, \quad \forall n\in\mathcal{N}_t,\ 1\leq t\leq T, \tag{6}$$

$$S_n\cdot\theta_n \geq 0, \quad \forall n\in\mathcal{N}_T, \tag{7}$$

$$\sum_{m\in\mathcal{A}(n)} e_m \leq 1, \quad \forall n\in\mathcal{N}_T e_n \geq 0, \forall n\in\mathcal{N}. \tag{8}$$

Therefore, we have the constraint

$$\sup -S_0\cdot\theta_0 + C_0e_0\psi_0 \geq v,$$

and constraints (6)–(8) equivalent to the constraint (5). Since we can omit the sup without changing the problem, and using the dual transformation in the objective function described above, we obtain the main result of the paper.

Theorem 4.5: *The buyer quantile hedge problem* $[PACP]$ *for an American claim* C *in discrete-time finite-state* *markets is posed as the problem* [*BPACP*] (*Bilinear PACP*)

$$
\begin{aligned}
&\min \sum_{n\in\mathcal{N}_T} \zeta_n,\\
&\text{s.t.} \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} \zeta_m \geq p_n\psi_n, \quad \forall n\in\mathcal{N}_T,\\
&-S_0\cdot\theta_0 + C_0 e_0\psi_0 \geq v,\\
&S_n\cdot(\theta_n-\theta_{\pi(n)}) = C_n e_n\psi_n, \quad \forall n\in\mathcal{N}_t,\ 1\leq t\leq T,\\
&S_n\cdot\theta_n \geq 0, \quad \forall n\in\mathcal{N}_T,\\
&\sum_{m\in\mathcal{A}(n)} e_m \leq 1, \quad \forall n\in\mathcal{N}_T,\\
&e_n \geq 0, \quad \forall n\in\mathcal{N},\\
&\psi_n \geq 1, \quad \forall n\in\mathcal{N},\\
&\zeta_n \geq 0, \quad \forall n\in\mathcal{N}.
\end{aligned}
$$

Furthermore, any optimal portfolio strategy θ^* *to BPACP solves problem ACP.*

Note that the problem BPACP involves only continuous variables! But it is a bilinear and hence non-convex problem (Bard 1998). Such problems can be notoriously difficult to solve numerically. On the other hand, we could have equally stated the above result as follows.

Theorem 4.6: *The buyer quantile hedge problem* [*PACP*] *for an American claim C in discrete-time finite-state markets is posed as the problem* [*BBPACP*] (*Bilinear Binary PACP*)

$$
\begin{aligned}
&\min \sum_{n\in\mathcal{N}_T} \zeta_n,\\
&\text{s.t.} \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} \zeta_m \geq p_n\psi_n, \quad \forall n\in\mathcal{N},\\
&-S_0\cdot\theta_0 + C_0 e_0\psi_0 \geq v,\\
&S_n\cdot(\theta_n-\theta_{\pi(n)}) = C_n e_n\psi_n, \quad \forall n\in\mathcal{N}_t,\ 1\leq t\leq T,\\
&S_n\cdot\theta_n \geq 0, \quad \forall n\in\mathcal{N}_T,\\
&\sum_{m\in\mathcal{A}(n)} e_m \leq 1, \quad \forall n\in\mathcal{N}_T,\\
&e_n \in \{0,1\} \quad \forall n\in\mathcal{N},\\
&\psi_n \geq 1, \quad \forall n\in\mathcal{N},\\
&\zeta_n \geq 0, \quad \forall n\in\mathcal{N}.
\end{aligned}
$$

Furthermore, any optimal portfolio strategy θ^* *to BBPACP solves problem ACP while the optimal* e^* *give the optimal exercise strategy.*

Proof: The proof is identical to that of the previous theorem since we could equally represent the buyer no-arbitrage value,

$$\inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}[C_\tau\psi_\tau],$$

using problem AP1 by corollary 4.4. □

It is known that some numerical algorithms and software exist for the numerical solution of the problems BPACP and BBPACP. On the other hand, as we shall see in section 5, an exact linearization of BPACP that uses the binary nature of the exercise variables e gives a mixed-integer linear formulation for which many well-developed algorithms and software are available. We treat this topic next.

4.4. An exact linearization

Consider the following linear mixed-integer program [LBPACP]:

$$
\begin{aligned}
&\min \sum_{n\in\mathcal{N}_T} \zeta_n,\\
&\text{s.t.} \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} \zeta_m \geq p_n\psi_n, \quad \forall n\in\mathcal{N},\\
&-S_0\cdot\theta_0 + C_0 w_0 \geq v,\\
&S_n\cdot(\theta_n-\theta_{\pi(n)}) = C_n w_n, \quad \forall n\in\mathcal{N}_t,\ 1\leq t\leq T,\\
&S_n\cdot\theta_n \geq 0, \quad \forall n\in\mathcal{N}_T,\\
&\sum_{m\in\mathcal{A}(n)} e_m \leq 1, \quad \forall n\in\mathcal{N}_T,\\
&e_n \in \{0,1\}, \quad \forall n\in\mathcal{N},\\
&w_n \leq M e_n, \quad \forall n\in\mathcal{N},\\
&w_n \leq \psi_n, \quad \forall n\in\mathcal{N},\\
&\psi_n \geq 1, \quad \forall n\in\mathcal{N},\\
&\zeta_n \geq 0, \quad \forall n\in\mathcal{N},
\end{aligned}
$$

where M can be chosen to be a suitable positive constant times v in our computational experience.

Theorem 4.7: $OPT(BPACP) = OPT(LBPACP)$.

Proof: Let ζ^*, ψ^*, Θ^*, e^* be feasible for BPACC. Assume $e^*\in E$ after recalling that we can equally solve BPACC as a mixed-integer nonlinear optimization problem following theorem 4.6. Define $w_n^* = \psi_n^* e_n^*$ for all $n\in\mathcal{N}$. For suitable M, this is feasible for LBPACC.

For the converse, assume you have ζ^L, ψ^L, Θ^L, e^L, w^L that are feasible for LBPACC. If for all $n\in\mathcal{N}$ we have $w_n^L = e_n^L\psi_n^L$, then ζ^L, ψ^L, Θ^L, e^L, w^L is clearly feasible for BPACC. On the other hand, if there exists $\eta\in\mathcal{N}$ such that $w_\eta^L \neq e_\eta^L\psi_\eta^L$ where $e_\eta^L = 1$, we have that $w_\eta^L \neq \psi_\eta^L$, i.e. $w_\eta^L < \psi_\eta^L$. Consider the corresponding constraint where w_η occurs on the right-hand side:

$$S_\eta\cdot(\theta_\eta^L - \theta_{\pi(\eta)}^L) = C_\eta w_\eta^L < C_\eta\psi_\eta,$$

i.e. we have

$$S_\eta\cdot(\theta_\eta^L - \theta_{\pi(\eta)}^L) + \alpha = C_\eta\psi_\eta^L,$$

for some $\alpha > 0$. But, since $S_n^0 = 1$ for all $n\in\mathcal{N}$ we can absorb the slack α by adding α to θ_η^L, i.e. $\theta_\eta^L = \theta_\eta^L + \alpha$. This operation does not affect the portfolio values in the nodes leading to node η, i.e. the nodes in $\mathcal{A}(\eta)\backslash\eta$. The impact on the portfolio positions in the descendant nodes $\mathcal{D}(\eta)$ is an increase in the net portfolio value, which propagates into the leaf nodes' portfolio position as a net increase and thus preserves feasibility. Hence, we have obtained a new set of portfolio positions, and hence a feasible solution to BPACC with objective function value identical to the objective function value of LBPACC at ζ^L, ψ^L, Θ^L, e^L, w^L. □

Let RLBPACC denote the linear programming relaxation of LBPACC.

Proposition 4.8:

$$\frac{OPT(LBPACC)}{OPT(RLBPACC)} \leq \frac{v}{\Pi^{\downarrow}(C)}.$$

Proof: It is clear that 1 is a lower bound on the optimal value of RLBPACC. On the other hand, $\psi = v/\Pi^{\downarrow}(C)$ is feasible in BPACC and hence in LBPACC with objective function value $v/\Pi^{\downarrow}(C)$. □

Remark 1: The lower bound equal to 1 for OPT(RLBPACC) is tight. To see this, consider the following numerical example with three time periods $t=0, 1, 2$. The market consists of a risky stock and a bond with zero interest rate. The stock trades at 10 at time $t=0$, and moves to one of the values in {20, 15, 7.5} at $t=1$ with equal probability. If it is valued at 20 at $t=1$ it moves to one of {22, 21, 19} at $t=2$. If it is valued 15 it moves to one of {16, 14, 13}. Finally from 7.5 it can move to one of {9, 8, 7}. All nine sample paths are equally probable. Assume an American option with strike equal to 11 is priced by a potential buyer. Setting up and solving the problem RLBPACC with $v=8/3$ and $M=3$, one obtains an optimal value equal to one.

5. Computational results with S&P 500 options

In this section we demonstrate that the models advocated in the previous sections can be solved numerically. For simplicity of exposition we use the basic pricing models of the previous sections with a slight modification involving the 'calibrated option bounds' model proposed by King *et al.* (2005).

5.1. Calibrated option bounds

In the setting of King *et al.* (2005), liquid options traded in the market are used for hedging purposes in addition to securities. These liquid options give the investor the possibility of forming buy-and-hold strategies in the hedging portfolio sequence. In other words, every liquid option can be bought or shorted by the investor at time zero with the purpose of hedging a contingent claim, and no intermediary trading is available for these options. Assuming there are K such liquid options, we denote them by G^k, $k=1,\ldots,K$. Bid and ask prices observed in the market at time 0 for option k are denoted by F^k_{b} and F^k_{a}, respectively, with the latter greater than or equal to the former. G^k_n is the payoff of option k at node n of scenario 3 and G_n is the vector of option payoffs at node n. Under these assumptions the buyer problem [BPACC] should be modified as [CBPACC]

(Calibrated Bilinear PACP)

$$\begin{aligned}
&\min \sum_{n\in\mathcal{N}_T} \zeta_n,\\
&\text{s.t.} \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} \zeta_m \geq p_n\psi_n, \quad \forall n\in\mathcal{N}_T,\\
&-S_0\cdot\theta_0 - F_a\cdot\xi_+ + F_b\cdot\xi_- + C_0e_0\psi_0 \geq v,\\
&S_n\cdot(\theta_n-\theta_{\pi(n)}) = G_n\cdot(\xi_+ - \xi\xi_-)C_ne_n\psi_n, \quad \forall n\in\mathcal{N}_t,\ 1\leq t\leq T,\\
&S_n\cdot\theta_n \geq 0, \quad \forall n\in\mathcal{N}_T,\\
&\sum_{m\in\mathcal{A}(n)} e_m \leq 1, \quad \forall n\in\mathcal{N}_T,\\
&e_n > 0, \quad \forall n\in\mathcal{N},\\
&\psi_n \geq 1, \quad \forall n\in\mathcal{N},\\
&\xi_+,\xi_- \geq 0,\\
&\zeta_n \geq 0, \quad \forall n\in\mathcal{N}.
\end{aligned}$$

where ξ_+, $\xi_-\in\mathbb{R}^K$ denote the long and short option portfolio position vectors, respectively.

The passage to the linearized, mixed-integer model can be accomplished as in the previous section.

5.2. Numerical results

We use 48 European options written on the S&P 500 index (table 1) in our computational experimentation. The option data were available in the market on September 10, 2002. The first 21 of the options are call options and the remainder are put options. Strikes and maturities as well as actual bid and ask prices (columns F_{b} and F_{a}) of these options are given in table 1. To create a test case close to reality we treat a given option C as an American option, while the remaining 47 options are taken as European options to be used in buy-and-hold policies in hedging C. We repeat this exercise for 14 of the options given in table 1, namely call options numbered 15–20 and those put options numbered 41–48.

We use a four period setting where we assume that investors can trade at days 0, 17, 37 and 100. We use $S=(1, S^1)$ as the traded securities. Having $S^0=1$ for all dates means that the interest rate is zero. We assume that the price of the S&P 500 index (i.e. S^1) follows a geometric Brownian motion. Under this assumption, we generate a scenario tree by the Gauss–Hermite process discussed by Omberg (1988) and King *et al.* (2005) in detail. The procedure works as follows. We assume that the value S^1 of the S&P 500 index evolves as a geometric Brownian motion with daily drift d and volatility σ. Let l be the length of period t in days. Then, the logarithm $\zeta_t = \ln S^1_t$ evolves according to

$$\zeta_t = \zeta_{t-1} + d_t + \epsilon_t,$$

where $d_t = l_t d$, and ϵ_t is normally distributed with zero mean and standard deviation $\sigma_t = \sqrt{l_t}\sigma$. Using given parameters ζ_0 and the initial values of ζ, l_t, $t=1,\ldots,T$, d and σ, we construct a scenario tree approximation to the stochastic process ζ_t using Gauss–Hermite quadrature as advocated by Omberg (1988), Pennanen and Koivu (2002)

Table 1. Option data.

Option No.	Type	Strike	Maturity	F_b	F_a
1	Call	890	17	31.5	33.5
2	Call	900	17	24.4	26.4
3	Call	905	17	21.2	23.2
4	Call	910	17	18.5	20.1
5	Call	915	17	15.8	17.4
6	Call	925	17	11.2	12.6
7	Call	935	17	7.6	8.6
8	Call	950	17	3.8	4.6
9	Call	955	17	3	3.7
10	Call	975	17	0.95	1.45
11	Call	980	17	0.65	1.15
12	Call	900	37	42.3	44.3
13	Call	925	37	28.2	29.6
14	Call	950	37	17.5	19
15	Call	875	100	77.1	79.1
16	Call	900	100	61.6	63.6
17	Call	950	100	35.8	37.8
18	Call	975	100	26	28
19	Call	995	100	19.9	21.5
20	Call	1025	100	12.6	14.2
21	Call	1100	100	3.4	3.8
22	Put	750	17	0.4	0.6
23	Put	790	17	1	1.3
24	Put	800	17	1.3	1.65
25	Put	825	17	2.5	2.85
26	Put	830	17	2.6	3.1
27	Put	840	17	3.4	3.8
28	Put	850	17	3.9	4.7
29	Put	860	17	5.5	5.8
30	Put	875	17	7.2	7.8
31	Put	885	17	9.4	10.4
32	Put	750	37	5.5	5.9
33	Put	775	37	6.9	7.7
34	Put	800	37	9.3	10
35	Put	850	37	16.7	18.3
36	Put	875	37	23	24.3
37	Put	900	37	31	33
38	Put	925	37	41.8	43.8
39	Put	975	37	73	75
40	Put	995	37	88.9	90.9
41	Put	650	100	5.7	6.7
42	Put	700	100	9.2	10.2
43	Put	750	100	14.7	15.8
44	Put	775	100	17.6	19.2
45	Put	800	100	21.7	23.7
46	Put	850	100	33.3	35.3
47	Put	875	100	40.9	42.9
48	Put	900	100	50.3	52.3

and King *et al.* (2005). The scenario tree generation procedure consists of using Gauss–Hermite quadrature to obtain a sample $(\epsilon_1^{i_1})_{i_1=1}^{\nu_1}$ of dimension ν_1 with associated positive probabilities $(\pi_1^{i_1})_{i=1}^{\nu_1}$. Hence, we obtain an approximation of possible values of the index at time $t=1$ using the equation

$$\zeta_1^{i_1} = \zeta_0 + d_1 + \epsilon_1^{i_1}, \quad i_1 = 1, \ldots, \nu_1.$$

For time period $t=2$ we generate a sample $(\epsilon_2^{i_2})_{i_2=1}^{\nu_2}$ of dimension ν_2 with associated positive probabilities $(\pi_2^{i_2})_{i_2=1}^{\nu_2}$ to obtain the possible values of the logarithmic index as

$$\zeta_2^{i_1,i_2} = \zeta_1^{i_1} + d_2 + \epsilon_2^{i_2}, \quad i_1 = 1, \ldots, \nu_1,\ i_2 = 1, \ldots, \nu_2.$$

Repeating this procedure for all time points up to time T, we obtain a scenario tree where the nodes $\mathcal{N}_t$ at time t are labeled by the t-tuple $(i_1, \ldots, i_t)$. In the notation of section 3, we have that the set $\mathcal{N}$ of all nodes in the tree is given as the union of all nodes for each time point t, i.e. $\mathcal{N} = \mathcal{N}_1 \cup \cdots \cup \mathcal{N}_T$. The parent node $a(i_1, \ldots, i_t)$ of $(i_1, \ldots, i_t)$ is the node labeled $(i_1, \ldots, i_{t-1})$; the child nodes $\mathcal{D}(i_1, \ldots, i_t)$ of the node $(i_1, \ldots, i_t)$ is the set $\{(i_1, \ldots, i_{t+1}) \in \mathcal{N}_{t+1} \mid i_{t+1} \in \{1, \ldots, \nu_{t+1}\}\}$. Finally, the probability distribution P for the leaf nodes is specified as $p(i_1, \ldots, i_T) = \pi_1^{i_1} \cdots \pi_T^{i_T}$, and $S_n = e^{\zeta_n}$ for all $n \in \mathcal{N}$. This completes the specification of the scenario tree. As the number of branches ν increases, the tree converges weakly to a discrete-time geometric Brownian motion, as shown by Pennanen and Koivu (2002).

We assume a branching structure of (50, 10, 10), which means that the tree divides into 50 nodes in the first period. Then, each node branches into 10 nodes in the second period, hence there are an additional 500 nodes in the third period. Then again each node of the second period is divided into 10 and there are 5000 leaf nodes of the tree. The resulting bilinear optimization models have 27,299 constraints and 37,299 variables. The linearized model has 32,850 constraints and 53,952 variables. We use $d=0.0001$, $\sigma=0.013175735$ and $S_0^1 = 909.58$, which was the closing value of the index on 10.9.2002.

We use the GAMS modeling language (Brooke *et al.* 1992) with the MOSEK (Mosek 2009) mixed-integer linear programming solver to solve the linearized version of CBPACC, and the nonlinear programming solver CONOPT (Drud 2007) to solve CBPACC with no guarantee of global optimality. We used the solvers with the default parameters. The results are reported in table 2. We give the results obtained by choosing the option price v as $1.05v^*$ and $1.1v^*$, respectively, where v^* denotes the no-arbitrage lower hedging price for the option in question. We singled out options with longer maturity (100 days) to make more interesting test cases, and therefore we report results with only a subset of the 48 options. In table 2 the CONOPT results are given under the heading 'BPACC', and those of MOSEK under 'LBPACC'. For all runs with the mixed-integer model we use $M=2v^*$. It appears that CONOPT is able to find an optimal or nearly optimal solution in several cases, and MOSEK is able to solve to optimality a majority of the test instances (to be precise 18 out of 28, i.e. approximately 65% of the test instances) in reasonable computing times. The cases indicated by an asterisk are the instances where either MOSEK was not able to find an optimal solution due to numerical difficulties (it stops with the best possible mixed-integer solution in that case) or a solution is reported with an objective function value higher than that given by CONOPT. The numerical difficulties are attributed to the potentially very small values of the probabilities calculated during the Gauss–Hermite-based scenario tree generation procedure described above. Further algorithmic research is needed to address these numerical issues.

Table 2. Numerical results with S&P 500 options using MOSEK and CONOPT solvers.

Option properties		Results for %5 Expanded prices				Results for %10 Expanded prices			
		BPACC		LBPACC		BPACC		LBPACC	
Type	Strike	Objective value	Time	Objective value	Time	Objective value	Time	Objective value	Time
Call	875	1.0177	268.50	19.26*	478.43	1.1543	155.12	2.40*	423.77
Call	900	1.0153	276.04	12.49*	507.38	1.0306	298.62	13.17*	562.34
Call	950	1.0042	458.61	1.0042	308.28	1.0123	396.01	1.0123	631.49
Call	975	1.0035	405.30	1.0034	146.50	1.0101	349.37	1.0101	1314.68
Call	995	1.0027	417.43	1.0027	41.00	1.0079	287.57	1.0079	1248.10
Call	1025	1.0002	316.85	1.0002	136.39	1.0010	301.88	1.0010	69.68
Put	650	1.0001	124.80	1.0001	155.97	1.0001	145.55	1.0002*	35.22
Put	700	1.00002	21.25	1.00003*	112.74	1.00002	22.01	1.0002*	31.88
Put	750	1.00002	82.06	1.00002	116.65	1.00002	82.47	1.00002*	967.43
Put	775	1.00003	103.19	1.00003	125.29	1.00005	133.31	1.00006*	22.91
Put	800		*Infeasible*			1.0001	583.10	1.00005	152.51
Put	850	1.0001	796.48	1.0001	151.05	1.0001	1266.20	1.0001	208.56
Put	875	1.0001	1020.71	1.00005	774.50	1.0001	1254.80	1.0001	1093.06
Put	900	1.0001	995.81	1.0001	775.39	1.0001	1751.11	1.0001	189.83

6. Conclusions

We have addressed the problem of quantile hedging for American contingent claims from the perspective of the buyer of a contingent claim in discrete-time financial markets. After a general exposition in a discrete-time infinite-state space setting we specialize our results to the finite-dimensional probability setting. The specialization resulted in finite-dimensional optimization problems which turn out to be bilinear or mixed-integer linear programming problems. We have shown that the problems can be processed numerically by state-of-the art solvers with default parameters for the case of finding buyer quantile hedge price bounds for S&P 500 options using other such options as part of the hedge portfolio.

Acknowledgements

The assistance of Tardu S. Sepin with the numerical experiments is gratefully acknowledged. The research reported in this paper was partially supported by TUBITAK grant 107K250.

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